ADDITION IN A CLASS OF NONLINEAR STIELTJES INTEGRATORS

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ABSTRACT

A product integral formula is established for the generation of an evolution system by the sum of two Stieltjes integrators. The class of Stieltjes integrators considered is the class recently introduced and studied by J. V. Herod.

I. Introduction

Let Y be a Banach space with norm N, and let R be an interval (bounded or otherwise) of real numbers. J. V. Herod [5] has recently introduced a class OA of order-additive Stieltjes integrators, a class OA of order-multiplicative evolution systems, and a solution transformation $\mathscr{E}$ from OA *onto* OM such that if V and M are in OA and OM, respectively, and $M = \mathscr{E}[V]$, then

$$M(a,b)[p] = p + \int_a^b V[M(\ , \ b)[p]]$$

whenever (a,b,p) is in $R \times R \times Y$ and $a \geqq b$. The class of nonlinear Stieltjes equations thus studied turns out to include a large class of nonautonomous differential equations and classes of autonomous differential equations similar to those studied by G. F. Webb [14] and R. H. Martin, Jr. [11].

Herod's continuity restrictions on members of OA are considerably weaker than Lipschitz conditions, and consequently the results of [5] constitute a generalization of similar results of J. S. MacNerney [9], [10] in which Lipschitz conditions were used. A natural question to ask, in either the Mac Nerney setting or the Herod setting, is the following: If each of V_1 and V_2 is in OA, can $\mathscr{E}[V_1 + V_2]$ be determined in terms of $\mathscr{E}[V_1]$ and $\mathscr{E}[V_2]$? This question has been

Received February 14, 1971 and in revised form August 17, 1971

answered in the MacNerney setting. In particular, in [6, th. 6] a sufficient condition is given for a multiplicatively integrated exponential identity to hold, in [7, ths. 5 and 7] this condition is found to be not only sufficient but also necessary and in [8, th. 4] it is found that in certain circumstances this idea can be extended to infinite sums and products. In the present work we shall indicate that these ideas extend into Herod's setting, i.e., Herod showed that if V is in OA then $\mathscr{E}[V]$ can be described by product integrals as

$$\mathscr{E}[V](a,b)[p] = \prod_a^b [I - V]^{-1}[p],$$

and herein it is shown that if each of V_1 and V_2 is in OA then so also is $V_1 + V_2$, and $\mathscr{E}[V_1 + V_2]$ can be described by product integrals as

$$\mathscr{E}[V_1 + V_2](a,b)[p] = \prod_a^b [I - V_1]^{-1}[I - V_2]^{-1}[p].$$

II. An exponential identity

We take OA, OM, and $\mathscr{E}$ to be as in [5]. Also, of course, properties iA and iM, for i in $\{1,2,3,4\}$, and all the results of [5], are assumed. The following theorem is the main result of this paper. Similar theorems have been found by H. F. Trotter [13] and P. R. Chernoff [1], [2] in linear semigroup theory, by B. W. Helton [3] in linear Stieltjes integral equations, and by J. L. Mermin [12] and G. F. Webb [15] in nonlinear semigroup theory.

THEOREM 1. *Suppose that each of V_1 and V_2 is in OA. Then $V_1 + V_2$ is in OA, and*

$$E[V_1 + V_2](a,b)[p] = \prod_a^b [I - V_1]^{-1}[I - V_2]^{-1}[p]$$

whenever (a,b,p) is in $R \times R \times Y$ and $a \geqq b$.

We shall take the notion of product integral, $\prod$, to be defined as in [10]. It is easy to see that if each of V_1 and V_2 is in OA then $V_1 + V_2$ satisfies each of 2A, 3A, and 4A. Furthermore, the results of Martin [11] can be used to show that $V_1 + V_2$ satisfies 1A and hence is in OA. This approach is unsatisfactory in that it leaves the formula in Theorem 1 unverified. Our approach is to embed Theorem 1 in a more general theorem which includes and extends portions of Herod's Main Theorem [5].

Let SI be the class to which U belongs only in case $U(a,b)$ is a function from (all of) Y to Y whenever (a,b) is in $R \times R$ and $a \geqq b$, and each of conditions C1, C2, C3, and C4 holds.

C1: There is a continuous nondecreasing real-valued function ρ on R such that if (a,b) is in $R \times R$ and $a \geqq b$ and $\rho(a) - \rho(b) \leqq \frac{1}{2}$ then $I - U(a,b)$ is a bijection and

$$N[[I - U(a,b)]^{-1}[p] - [I - U(a,b)]^{-1}[q]] \leqq N[p - q] \exp[2\rho(a) - 2\rho(b)]$$

whenever (p,q) is in $Y \times Y$.

C2: If (a,b) is in $R \times R$ and $a \geqq b$ then there is a nondecreasing real-valued function γ on $[b,a]$ such that, if p is in Y and ε is a positive number, there is a positive number d such that $N[U(x,y)[p] - \Sigma_{k=1}^{n} U(t_{k-1},t_k)[p]] \leqq \varepsilon[\gamma(x) - \gamma(y)]$ whenever $a \geqq x \geqq y \geqq b$ and $x - y < d$ and $\{t_k\}_{k=0}^{n}$ is a chain from x to y.

C3: If (a,b) is in $R \times R$ and $a \geqq b$, and B is a bounded subset of Y, then there is a continuous nondecreasing function α on $[b,a]$ such that $N[U(x,y)[p]] \leqq \alpha(x) - \alpha(y)$ whenever $a \geqq x \geqq y \geqq b$ and p is in B.

C4: If (a,b) is in $R \times R$ and $a \geqq b$ then there is a nondecreasing real-valued function β on $[b,a]$ such that, if p is in Y and ε is a positive number, there are positive numbers d and δ such that $N[U(x,y)[p] - U(x,y)[q]] \leqq \varepsilon[\beta(x) - \beta(y)]$ whenever $a \geqq x \geqq y \geqq b$ and $x - y < d$ and q is in Y and $N[p - q] < \delta$.

Note that SI includes OA. Furthermore, the primary difference between SI and OA is the difference between 2A and C2, i.e., members of SI need only approximate order-additivity, they need not be order-additive.

THEOREM 2. *Let U be in SI. Then each of* (i), (ii), *and* (iii) *is true.*

(i) *If (a,b,p) is in $R \times R \times Y$ and $a \geqq b$ then*

$$\prod_{a}^{b} [I - U]^{-1}[p]$$

exists.

(ii) *If M is given by $M(a,b)[p] = \prod_a^b [I - U]^{-1}[p]$, then M is in OM.*

(iii) *If M is as in* (ii), *and V is in OA and $\mathscr{E}[V] = M$, then $V(a,b)[p] = \Sigma_a^b U[p]$ whenever (a,b,p) is in $R \times R \times Y$ and $a \geqq b$.*

Theorem 2 can be proved with a sequence of lemmas and theorems almost identical to those in [5, §2]. Most of the effort in [5, §2] is to show that if V is in OA then property 4A holds uniformly on compact sets and then to show the existence of a chain such that 4A holds uniformly on all partial products over refinements of that chain. In our present setting, similar results can be obtained with respect to not only condition C4 but with respect also to condition C2. The first use which Herod makes of order-additivity in [5, §2] in in the proof of [5, th. 2.1]. To prove part (i) of Theorem 2, one should proceed as Herod does in [5,§2], and bypass Herod's use of order-additivity with C2 and the appropriate uniformity conditions. Now (ii) of Theorem 2 follows from an argument almost identical to that for [5, th. 2.2]. An argument similar to that for [5, lemma 4.1], with appropriate use of C2, shows that if (a,b,p) is in $R \times R \times Y$ and $a \geqq b$, then $N[\Sigma_t U[p] - \Sigma_t[M-I]\ [p]]$ goes to zero over refinements of chains t from a to b. Since $V(a,b)\ [p] = \Sigma_a^b[M-I]\ [p]$, part (iii) of Theorem 2 follows immediately. We are now able to prove Theorem 1.

Proof of Thorem 1. Let each of V_1 and V_2 be in OA. Let U be such that

$$U(a,b)[p] = V_1(a,b)[p] + V_2(a,b)[I - V_1(a,b)][p]$$

whenever (a,b,p) is in $R \times R \times Y$ and $a \geqq b$. First we show that

$$\sum_a^b U[p] = V_1(a,b)[p] + V_2(a,b)[p]$$

or equivalently,

$$\sum_a^b V_2[I - V_1][p] = V_2(a,b)[p]$$

whenever (a,b,p) is in $R \times R \times Y$ and $a \geqq b$. Let (a,b,p) be in $R \times R \times Y$ with $a \geqq b$. Let β_2 go with V_2 as in 4A. Let B be the bounded set consisting of the single element p, and let α_1 go with V_1 as in 3A. Let ε be a positive number. Find a positive number δ such that $N[V_2(x,y)[q] - V_2(x,y)[p]] \leqq \varepsilon[\beta_2(x) - \beta_2(y)]$ whenever $a \geqq x \geqq y \geqq b$ and q is in Y and $N[p-q] < \delta$. Find a positive number d such that $\alpha_1(x) - \alpha_1(y) < \delta$ whenever $a \geqq x \geqq y \geqq b$ and $x - y < d$. Let $\{t_k\}_{k=0}^n$ be a chain from a to b such that $t_{k-1} - t_k < d$ whenever k is an integer in $[1,n]$. Now

$$\begin{aligned} N[V_2(a,b)[p] - \sum_t V_2[I - V_1][p]] &\leqq \sum_{k=1}^n N[V_2(t_{k-1},t_k)[p] \\ &\qquad - V_2(t_{k-1},t_k)[I - V_1(t_{k-1},t_k)]\ [p]] \\ &\leqq \sum_{k=1}^n \varepsilon[\beta_2(t_{k-1}) - \beta_2(t_k)] \\ &= \varepsilon[\beta_2(a) - \beta_2(b)], \end{aligned}$$

so

$$\sum_a^b V_2[I - V_1][p] = V_2(a,b)[p].$$

Now if each of A, B, and C is a function from Y to Y, and each of $I - A$, $I - B$, and $I - C$ is a bijection, then $C = A + B[I - A]$ only in case $[I - C]^{-1} = [I - A]^{-1}[I - B]^{-1}$, so to complete the proof it suffices to show that U is in SI. Recall that if c is a number in the interval $[0, \frac{1}{2}]$ then $[1 - c]^{-1} \leqq 1 + 2c \leqq \exp[2c]$. Thus, if ρ_1 and ρ_2 go with V_1 and V_2, respectively, as in 1A, and ρ is a function such that $\rho(a) - \rho(b) = \int_b^a |d\rho_1| + \int_b^a |d\rho_2|$ whenever (a, b) is in $R \times R$ and $a \geqq b$, then U satisfies C1 with respect to ρ.

Note that C2 follows from the computation above. Now suppose that B is a bounded subset of Y and (a, b) is in $R \times R$ with $a \geqq b$. Find a number d_1 such that $N[p] \leqq d_1$ whenever p is in B and a number d_2 such that $N[V_1(x, y)[p]] \leqq d_2$ whenever $a \geqq x \geqq y \geqq b$ and p is in B (this can be done by 3A). Find continuous nondecreasing functions α_1 and α_2 on $[b, a]$ such that $N[V_i(x, y)[p]] \leqq \alpha_i(x) - \alpha_i(y)$ whenever $a \geqq x \geqq y \geqq b$ and p is in Y and $N[p] \leqq d_1 + d_2$ and i is in $\{1, 2\}$. Now let $\alpha = \alpha_1 + \alpha_2$, and C3 is verified. Now let (a, b) be in $R \times R$ with $a \geqq b$, let p be in Y, and let ε be a positive number. If i is in $\{1, 2\}$, let β_i go with V_i as in 4A. Let B be the set to which q belongs only in case q is in Y and $N[p - q] \leqq 1$, and let α_1 go with V_1 as in 3A. Find a positive number δ such that $\delta < \frac{1}{2}$ and such that $N[V_i(x, y)\,[p] - V_i(x, y)\,[q]] \leqq \varepsilon[\beta_i(x) - \beta_i(y)]$ whenever $a \geqq x \geqq y \geqq b$ and q is in Y and $N[p - q] < 2\delta$ and i is in $\{1, 2\}$. Now find a positive number d such that $\alpha_1(x) - \alpha_1(y) < \delta$ whenever $a \geqq x \geqq y \geqq b$ and $x - y < d$. Suppose that $a \geqq x \geqq y \geqq b$, that $x - y < d$, that q is in Y, and that $N[p - q] < \delta$. Now

$$\begin{aligned}
&N[V_1(x, y)[q] + V_2(x, y)[I - V_1(x, y)][q] \\
&\quad - V_1(x, y)[p] - V_2(x, y)[I - V_1(x, y)][p]] \\
&\quad \leqq N[V_1(x, y)[q] - V_1(x, y)[p]] \\
&\qquad + N[V_2(x, y)[I - V_1(x, y)][q] - V_2(x, y)[p]] \\
&\qquad + N[V_2(x, y)[I - V_1(x, y)][p] - V_2(x, y)[p]] \\
&\quad \leqq \varepsilon[\beta_1(x) - \beta_1(y)] + \varepsilon[\beta_2(x) - \beta_2(y)] + \varepsilon[\beta_2(x) - \beta_2(y)].
\end{aligned}$$

Thus C4 is satisfied with $\beta = \beta_1 + 2\beta_2$, and the proof is complete.

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